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THE CREATIVITY ISSUE

EXAMINING MATHEMATICS CREATIVE THINKING THROUGH PICK'S THEOREM

By: Prof. Shahbazi (Associate Professor; MSLC Coordinator)

We all know how to find the area of simple shapes like rectangles, but what if we needed to find the area of a shape like shown in Figure 1? How would we do it? Is there a simple formula to calculate the area of shapes like this? Before I reveal a simple and interesting formula that could be used to calculate this shape's area, let's think about the concept of creativity in general for a moment.

We can try to understand creativity through Edward Bono's "Six Hats" thinking model [1]. According to Bono, one can break down the process of thinking into six stages and represent each stage with a coloured hat. We can imagine ourselves constantly changing these hats until we find an appropriate solution to a given problem.

In this model, the blue hat is responsible for defining a problem clearly, time management, the flow of ideas, and assisting in communication between different hats. The hat is blue like the sky, which demonstrates its leading role. Some questions that we might ask while wearing this hat could be: "what is the problem that we are facing? What are our goals and desired outcomes?"

The white hat looks like a blank piece of paper, ready to record whatever data our inquiry uncovers. The types of questions that we might ask while wearing this hat might be along the lines of: "what do I know and what don't I know about this problem?"

The yellow hat represents the initial optimistic stage of thinking about a problem. It provides us with a roadmap to our ultimate goal, and gives us the motivation to work towards it. It deals with questions like: "what are the benefits and potential positive outcomes of our proposed solutions?"

The black hat represents the pessimistic stage of thinking, which is every bit as necessary as the optimistic stage, because it can help us to avoid making mistakes as we carry out our solutions. The types of questions we might expect from the black hat might include: "what are some possible flaws of this way of thinking? What are the drawbacks of our method?"

The red hat is all about emotion and intuition. While wearing this hat, we are concerned with internal conflicts, which is why the hat is the colour of blood. The types of questions that we might ask at this stage could include "is this the right approach? Intuitively, what seems to be the best way to solve this problem?"

Finally, the green hat is concerned with creativity itself, which is why it bears the colour we most associate with the natural world. While wearing this hat, we strive to bring new methods and ideas into the foreground, in a way that surprises the other hats. While wearing this hat, we might ask questions like: "could we solve this problem more easily using another method? What other approaches might we consider?"

According to Tony Buzan, we can understand creativity through mind maps which model our thinking

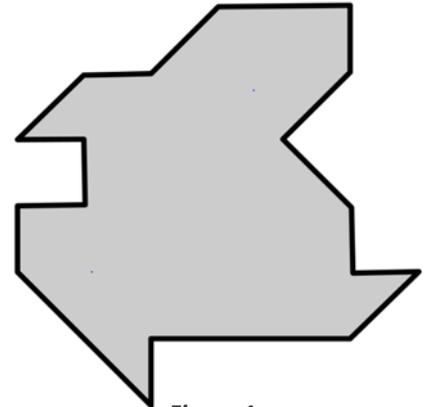


Figure 1

processes. The primary way that our brains learn is by association and connection. The way a central idea connects to related concepts shapes how we think about a subject. In order to generate new ideas, we need to make more and more connections between diverse ideas, just like a mind map. Creativity is a conscious and associative process. Our minds' potential for generating new ideas is all but limitless, since the possibilities of connecting details to one another are infinite. The image below describes creativity by combining the Six Hat and Mind Map thinking models.

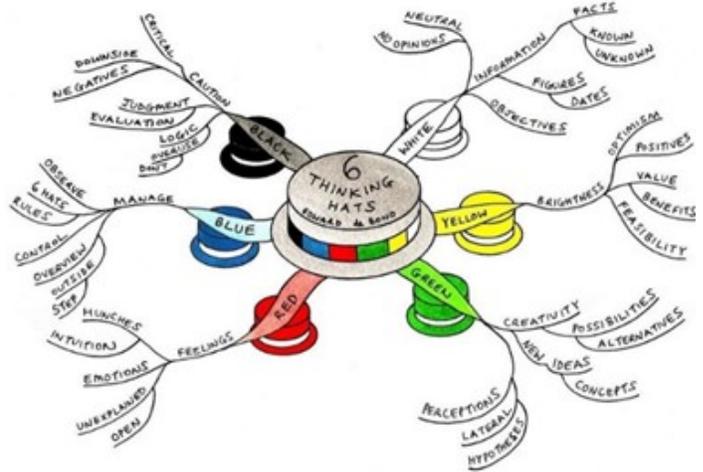


Figure 2 [2]

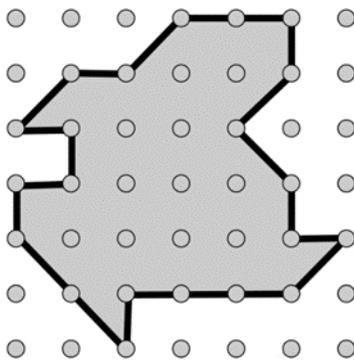


Figure 3

Now let's return to our original problem. I suggest that you cover the image with a collection of regularly spaced points located on vertices of the image as seen in Figure 3. If you can do this successfully, then the area can be calculated with Pick's formula for simple lattice polygons [3]. A simple polygon has no holes, and no two edges intersect. According to Pick's formula, the area of a simple lattice polygon is determined by the number of lattice points that lie on its boundary and its interior. If I denotes the number of lattice points in the interior, and B denotes the number of lattice points on the boundary, then:

$$A(S) = I + \frac{1}{2}B - 1.$$

All we have to know is the number of points inside and on the boundary of the image to be able to calculate the area of many planar shapes with strange looking boundaries.

In our case, we have 11 points inside and 21 points on the boundary. Here, I assume the distance between any two points in Figure 3 is one centimeter. So, by Pick's formula the area is:

$$A(S) = 11 + \frac{1}{2} \cdot (21) - 1 = 20.5 \text{ cm}^2$$

Does this formula surprise you? I cannot help but appreciate the beauty of this formula. In fact, the field of mathematics is full of mind-stretching problem solving tools such as Pick's formula. During the mathematical problem solving process, we connect different ideas and facts to produce original results. This reveals the deep connection between mathematical thinking and creativity.

[1] Buzan, T., "Make the Most of Your Mind," London: Pan Books, 1988.Rev.ed.

[2] <http://www.mindmapinspiration.com/category/creativity/>

[3] Weisstein, E., "Simple Polygon," MathWorld--A Wolfram Web Resource, [Online]. Available: <http://mathworld.wolfram.com/SimplePolygon.html>.



PLAYING CARDS



By: Olivia Rennie (Statistics TA, MSLC)

Playing cards have been a source of entertainment for centuries, used not only for games, but also for magic. The modern 52 card deck, which consists of four 13-card suits (diamonds, hearts, clubs, and spades), is estimated to have come into existence in the mid-to-late 1300s, and remains the most popular type of deck. While at first glance it might appear as though magic and mathematics are unrelated, the unique properties this remarkable pastime game possesses have allowed a multitude of tricks to evolve, many of which rely primarily on math and probability.

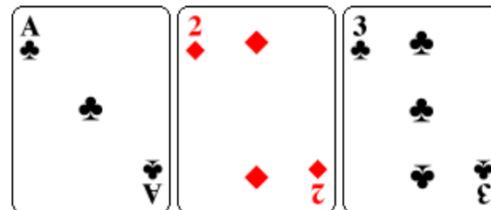
While there are many tools an aspiring magician needs to master to become proficient in card magic, a number of tricks can be successfully performed without relying on challenging hand manipulation techniques. Instead, the built-in mathematics playing cards possess enable beginner and expert magicians alike to complete tricks that will not only astound their audience members, but also prove to themselves that math can be extremely fun, and even downright magical!

To get you going on your journey into the realm of numbers and illusion, give the following tricks a shot with your family and friends, and prepare to be amazed!

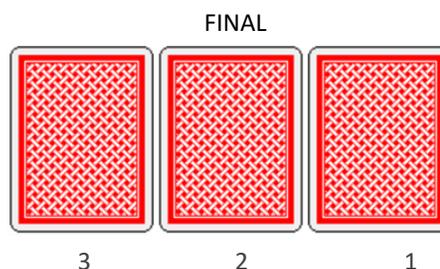
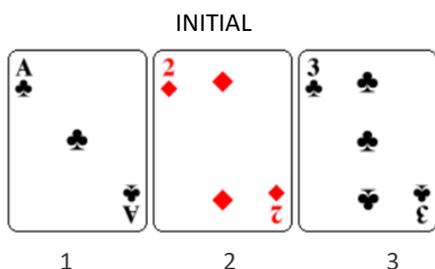
Trick 1: Hummer's One-Two-Three Trick

Effect: Place any Ace, 2 and 3 (one of each, any suit) face up in a row on the table, in that order. Turning away so you are unable to see the cards, ask a spectator to mentally pick a card and turn it over, also turning over the other two cards after switching their positions. Then turn around, collect the cards, and cut the pack repeatedly. Laying

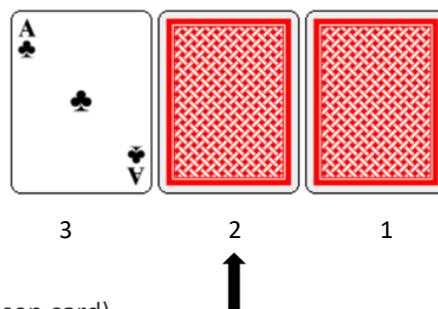
them face down on the table again, tell your spectator to choose the card that they think is theirs; if they guess incorrectly, tell them it is your turn to guess from the remaining two. The spectator then turns over the card they believe is theirs. If they are correct, congratulate them. If not, you will (using the method below) correctly turn over their card, and amaze your audience!



Method: When the spectator tells you that they have finished selecting their card, flipped it over, and switched the positions of the other two cards and also flipped them over, pick up the cards so that from **top to bottom you have the rightmost card, then the middle card, and then the leftmost card**. Cut the cards to the bottom, one or two at a time, in a manner that appears to have no pattern, when in reality you are keeping count. Stop once you have moved **10, 13, or 16** to the bottom. Now deal out the **top card to the middle, the second card to the right, and the third card to the left**. At this point, you must adjust your mental image so that instead of seeing the positions as being 1→2→3 from left to right, the positions are now 3→2→1 from left to right (a reversal of the earlier placements). Invite your spectator to turn over whichever card they believe is theirs. This is going to be considered position i . If it is their card, congratulate them for a good guess. If it isn't their card, then this incorrect pick will be considered a card with value j not equal to i in position i . Then turn over the card in the position k not equal to i or j , which is the correct choice.



As an example, let's say that they believe their card is the leftmost card, which is considered to be in position 3. When they flip it over, it is the Ace, and your spectator indicates that they guessed incorrectly. The correct card will be the middle card, which is in position **two**. This is derived as noted above. An Ace is considered value 1 (i.e. $i=1$). The rightmost card is in position $i=1$, and **so we turn over the card that is NOT in this position**, which is $k=2$; (position 2, or the middle position); you essentially take the value of the card, and turn over the card that is NOT in the position with that value—this will be your spectator's chosen card).



Mathematics: This trick is all about permutations in the symmetric group on $\{1,2,3\}$, which represent the initial positions of the Ace, 2, and 3. The spectator's actions result in an unknown transposition τ . The manner in which you pick up the cards will reverse their order (i.e. by picking them up right to left)—this is equivalent to the permutation $(1\ 3)$, another transposition. The cutting of $s = 1 \pmod{3}$ cards to the bottom of the pack comes down to the 3-cycle $(1\ 3\ 2)$, and the final placement of the cards on the table (disregarding the left-right switch for now) is the same as the transposition $(1\ 2)$. Since $(1\ 2)(1\ 3\ 2)(1\ 3)\tau = \tau$, using the convention that permutations are multiplied from right to left, we see that knowing the identity of just one card reveals what τ is: the left-right switch is just to make it less obvious!

Source: This is an invention of Bob Hummer's, as adapted by Max Abrams. This trick appears in Bob Longe's *Easy Magic Tricks* (Sterling, 1994), where Martin Gardner is also credited.

Trick 2: The Australian Shuffle

Effect: Ask a spectator to select a random number between 10 and 30. Count out that many cards from the deck and hand them to the spectator, who will do the “Australian Shuffle” (AKA The Down Under Deal) until one card remains. You correctly name that card and wow those watching!

The “Down Under” Deal

This is a technique that can be used in a variety of card tricks. Essentially, by the end of the deal, you will be left with a single card in your hand.

Holding the deck of cards, pick up the top card and set it face **down** on the table. Take the next card on top and place it **under** all the other cards (i.e. at the bottom of the deck, still face down). Continue doing this (take the top card and place it face down on the table, then take the new top card and place it on the bottom of the deck) until you are left with only one card in your hand, and the rest face down on the table.

Method: You **must know the top card on the deck at the outset**. There are many methods for doing this in a nonchalant way, so play around with it. Next, take the number your spectator chose, and (in your head) **compute twice the difference between the number they have selected and the highest power of two which is strictly less than it** (e.g. if fourteen is selected, compute $2*(14-8) = 12$, since $2^3 = 8$ is the highest power of two that is less than the chosen number (14)). Next, deal out this value of cards (in this example, 12 cards); as you deal them out, this will reverse the order, such that the top card is now on the bottom. When you get to the end, pick up the cards you have dealt out, and pretend you have a poor memory, asking the spectator to repeat their number (“14”). Thank them for correcting you, scoop an extra two cards off the deck (or however many additional cards you need to reach their chosen value), placing them **beneath the cards in your hand**. What you have done here is control the top card to (in this example), the 12th position in a packet of fourteen cards.

If the spectator calls out a number that is an exact power of two (e.g. 16), then simply deal that number into a pile.

In either case, hand the packet of cards to the spectator (face down so they can't see the value of the bottom card!). Direct them on how to do the down under shuffle. The last card they are holding is guaranteed to be original top card.

Mathematics: Things become clearer if we convert to base 2. Suppose that the number your spectator selects is $1abc\dots e$ in base 2. An exercise you can try is to show that the down under deal from a packet of $1abc\dots e$ cards always has the final card as that which started in position $abc\dots e0$, unless the number of cards in the packet is *exactly a power of two*; in that case, the down under deal ends up with the original bottom card.

Now, if $1abc\dots e$ is *not* a power of two, the highest power of 2 strictly less than it is $1000\dots 0$, and when this is subtracted from $1abc\dots e$ we get $abc\dots e$. Twice that is $abc\dots e0$. On the other hand, if $1abc\dots e = 1000\dots 0$ is a power of two, the next highest power of two is $100\dots 0$ (one fewer zero), the difference is also $100\dots 0$, and twice the difference gets back the total number of cards in the packet.

Either way, the Down Under Deal will end up with the top card you peeked at before starting the trick!

Source: Martin Gardner mentions learning the basic idea behind this trick from Mel Stover in *The Unexpected Hanging and Other Mathematical Diversions* (Simon and Schuster, 1969, republished by Univ. of Chicago Press, 1991), and credits magician John Scarne with the above presentation. He also mentions a 1939 Bob Hummer version, and describes several variations on the same theme due to Sam Schwartz and others.

Reference: These tricks, among others, may be found in an article created by Colm Mulcahy, of the American Mathematical Society (last updated 2000). <http://www.ams.org/publicoutreach/feature-column/fcarc-mulcahy1>

THE ROCHESTER MATH OLYMPIAD

By: Parker Glynn-Adey

On Saturday February 9th 2019, UTSC competed in the Rochester Math Olympiad for the first time. The Olympiad is an approachable math contest, and a great introduction to recreational mathematics. The questions are short, snappy, and anyone can make a little bit of progress on at least on problem. This year, UTSC students: Zijie Wang, Matthew Tranov, Minyi Ma, and Schinella D'Souza all wrote the contest, and placed our little team placed well among the 76 participants.

The Math Club hosted a series of four training sessions leading up to the Olympiad. Contest mathematics can stretch your math muscles, and build up resilience and grit. We encourage you to try out these problems, you might be surprised by the answers. Here are some questions from our training sessions.

The Olympiad had four questions, and we've selected the most approachable questions for you:

Q1) Which is bigger: 1000^{2000} or $2000! = 2000 \times 1999 \times 1998 \times \dots \times 3 \times 2 \times 1$?

Q2) The King of Calculus has ordered 1000 barrels of wine for his upcoming birthday party. However, an hour before the guests are set to arrive, his chief spy alerts him that the nefarious Duke of Algebra has poisoned one of the barrels with a special poison that will kill anyone who drinks it in precisely one hour, but is otherwise undetectable. Ten of the king's loyal servants volunteer to find the poisoned barrel by tasting the wine. Assuming these servants can get organized and drink any amount of wine instantaneously, will the king be able to determine which barrel contains the poison before the guests arrive? (Written by Matt Sourisseau)

Q3) Let $H = \{1, 2, 3, \dots, 2019\}$ and $P(H)$ be a set of all subsets of H (also known as the power set of H). If $f: P(H) \rightarrow \mathbb{R}$ is a function which satisfies $f(A \cap B) = \min\{f(A), f(B)\}$, for all $A, B \in P(H)$, find the maximum number of distinct values f can take. Above, $A \cap B$ denotes the intersections of subsets A and B .

Q4) Let n and m be positive integers. $n+m$ people are waiting in line at a box office. n of them have only \$5 bill each, while the rest possess only a \$10 bill each. The cost of a ticket is \$5. When the box office opens, there is no money in the cash register. If every person buys exactly one ticket, what is the probability that none of them will have to wait for change?

Everyone curious about writing the Rochester Math Olympiad is encouraged to contact Parker Glynn-Adey (parker.glynn.adey@utoronto.ca) for more information. Writing the contest is completely free, and a pizza lunch is provided afterwards. The UTSC Math Club will hold training sessions in early January 2020. Drop by the Math & Statistics Learning Centre for more information.

MSCL'S NEW VIRTUAL TUTORING TOOL: MC²

(Mathematics Classroom Collaborator)

By: Prof. Kang (Associate Professor; MSLC Statistics Coordinator)

Did you know that the MSLC offers a new virtual tutoring platform called MC² (Mathematics Classroom Collaborator)? Of the 117 students who took part in the recent MSLC survey, 54.7% said that they had never even heard of it. The purpose of this article is to promote MC² and explain how it works.

According to a study by Helvie-Mason (2012), 81.1% of post-secondary students wished that their professors offered more virtual communication options, and 83.8% would use app-based mobile communication if it was offered. Similarly, students at UTSC would like to be able to receive assistance from MSLC TA's after hours. Developing a real-time, chatroom with mathematical expressions functionality for novice users (i.e., non-TeX users) was not an easy task. Another critical requirement was that the software work on a variety of different platforms (tablets, smartphones, and laptop computers), which may run on a variety of different operating systems, such as Windows, MacOS, iOS, and Android. To ensure that students would have easy access to the software without having to pre-install anything, a browser-based architecture was appropriate. Possible input methods could include keyboard, mouse, stylus, touch screen, and camera. A web-based application would also lower the barrier to entry for first-time users.

On the surface, MC² presents itself as a familiar texting application, with the addition of mathematical capabilities. We have numbered its functions from (1) to (6) in Figure 1 below.

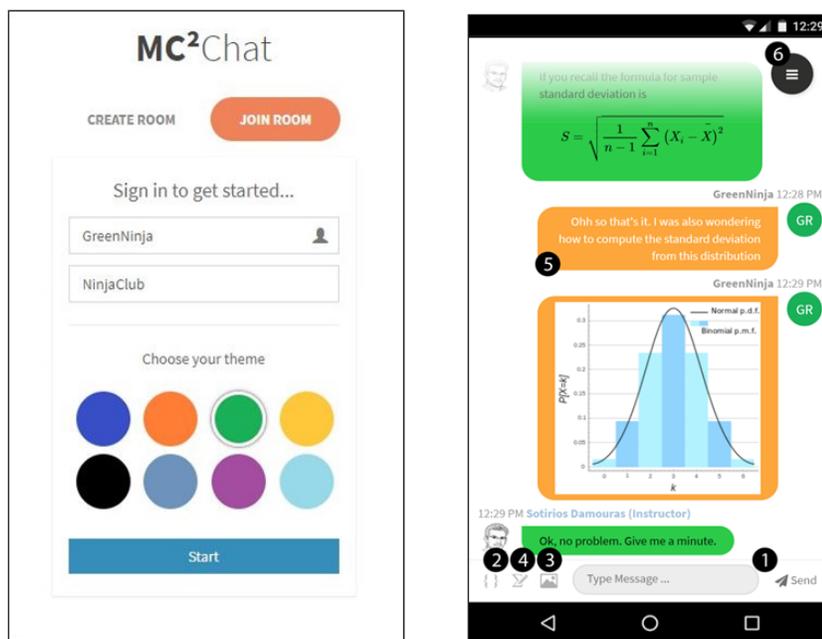


Figure 1: Right Panel: MC² Chat Screen (1) Text input field. (2) Insert TeX into text field (3) Input image from camera or gallery (4) Launch the mathematical expression editor. (5) Dialogue Pane: clicking on an image or mathematical expression launches the expression editor for annotation or modification. (6) Chat Option – tab to expand the options of the Online Users and Logout buttons. Left Panel: Initial Login Screen.

- ① Text Field: In addition to typing regular texts, the text field supports the insertion of TeX.
- ② TeX Editor: TeX input requires a number of symbols such as \$, \, {, }, _ and ^ . On smartphone keyboards, these symbols are often difficult to access quickly. In MC², these symbols can be found under this button.
- ③ Image/Camera: The image button allows the user to upload an image. The user can write mathematical expressions on a piece of paper, or take a picture with their smart phone and upload it, which could be convenient for those accessing the application on a smartphone. Students can upload screenshots of questions from their homework assignments or online lecture notes.
- ④ Mathematical Editor: Complex mathematical expressions can be entered using the mathematical editor, where equations can be inputted using a diagrammatic equation editor for non-TeX users. The symbols can be selected, moved, and resized based on the diagram editor UI principles. Diagrams can be drawn and recognized using a baseline structured approach and converted to TeX, which can be displayed on the messenger tab.
- ⑤ Dialogue Pane: The dialogue pane is where the conversation is displayed in chronological order. It is nearly identical to the panes found in text messaging apps. Instructors can delete posted messages and users can copy messages and paste them into the TeX editor.
- ⑥ Chat Options: click to expand the chat options. There are two options: The Online User button and the Logout button. The Online User button displays the user names or the users that are currently in the chatroom.

There are three easy steps to use this great tool:

1. Check the virtual tutoring office hours offered at the MSLC via our website (<https://www.utoronto.ca/mslc/virtual-tutoring>)

2. Click the virtual tutoring icon on the right side.

3. Provide “username” as your “UTORid” (not a student number!) and click “Login”.

Now, you can ask questions directly to your virtual tutoring TA in the text message field, and if you are using a smart phone, you can take a picture of your questions and upload it (push the image icon at the bottom of the text message field). An instructional video about MC² will be developed soon. We hope that you enjoy this service, and don't forget to leave your comments in our next MSLC survey!

Reference:

Helvie-Mason, L. (2012). Office Hours: `There's an App for that?!': Student perceptions of faculty channels for out-of-class communication. International Journal of Instructional Technology and Distance Learning, 9.

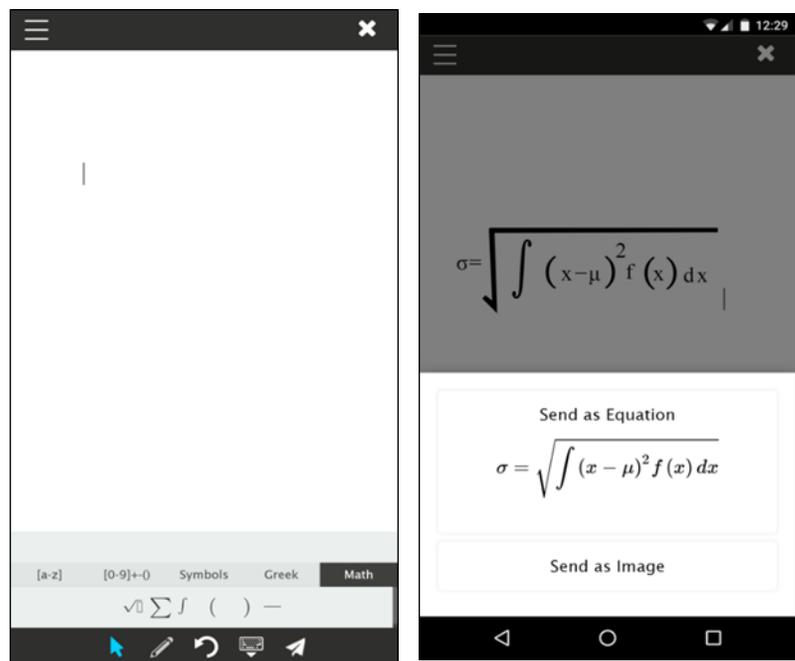


Figure 2: MC² Mathematical Editor – non-TeX users can click on math symbols or Greek letters from the menu and make a diagrammatic equation that can either be converted to TeX expressions and sent as a text file, or sent directly to the dialogue pane as an image file.

Euler and the partition numbers

By: David Salwinski, Mathematics Student

Suppose you have six identical poker chips. How many ways are there to stack them up? Well, you could have one stack 6 chips high, or a stack 5 chips high with 1 chip lying separate, or a stack of four chips and a stack of 2 chips, or two stacks of 3 chips, etc. There are eleven possibilities in total, and they can be represented abstractly by the sums below.

6	4 + 1 + 1	2 + 2 + 1 + 1
5 + 1	3 + 2 + 1	2 + 1 + 1 + 1 + 1
4 + 2	2 + 2 + 2	1 + 1 + 1 + 1 + 1 + 1
3 + 3	3 + 1 + 1 + 1	

This shows that our question about stacking poker chips is equivalent to finding the ways in which the integer 6 can be expressed as a sum of smaller positive integers. In mathematics, such a representation is called a *partition* of an integer. Partitions appear in various problems in combinatorics, like in counting the possible arrangements of tiles of various sizes or in certain change-making problems, such as finding the ways to make change for a dollar using lesser coins of different denominations.

In applications, it is often necessary to restrict our attention to specific types of partitions. For instance, *even partitions* and *odd partitions* are those partitions that use only even or odd numbers throughout, while *distinct partitions* are those that do not contain duplicates of numbers. In the example above, notice that there are three even partitions, four odd partitions, and four distinct partitions of 6. The study of partitions raises many natural questions to the mathematician, such as whether there is a formula for the number of partitions (of a certain type) of a given number n , how quickly the number of partitions grows as n grows, or if there are any patterns to be found within the lists of possible partitions.

In 1740, the French mathematician Philippe Naudé had been working on a problem involving certain kinds of partitions when he wrote to Leonhard Euler, the preeminent mathematician of the time, asking for insight. Within several days Euler had responded to Naudé, including in his letter a proof of the surprising fact that given any positive integer, its number of odd partitions is exactly equal to its number of distinct partitions. More impressive than the statement of this new theorem is the manner in which it was proved: Euler had produced the first example of a proof using objects now known as generating functions, which turn out to be very useful in solving all kinds of combinatorial problems and are a hot topic of research even to this day.

Euler's Theorem: For any positive integer n , the number of odd partitions of n is equal to the number of distinct partitions of n .

Proof: Euler began by introducing the following function, defined for $-1 < x < 1$.

$$f(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5) \dots$$

As intimidating as that infinite product might look, it is a function just like any other, having a unique output for any given input from its domain. Euler then considered what happens when the infinite product is expanded.

Progressing in order of degree, we see there is only one way to obtain a degree zero term, and it comes from

multiplying all the 1s within each bracket with each other. There is also only one way to obtain a degree one term, and it comes from multiplying the x in the first bracket by all the other 1s that follow in the other brackets. Continuing on, we find that $(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\dots = 1+x+x^2+2x^3+2x^4+3x^5+4x^6+\dots$

The coefficients appearing in the expansion count the number of ways of obtaining a specific power of x by multiplying together smaller powers of x . For example, there are four different ways to obtain an x^6 term: it can come from the lone x^6 term in the product (multiplied by all the 1s in the other brackets), or from multiplying the x term by the x^5 term (and then multiplying that by all the 1s), or from multiplying an x^2 by an x^4 , or from multiplying an x and an x^2 and an x^3 together. The powers of these terms are exactly the numbers added together to obtain the four distinct partitions of 6! So, the coefficient of the x^n term in our expansion will be the number of distinct partitions of n , which we may abbreviate by $D(n)$. Our function can thus be written compactly as

$$f(x) = \prod_{n=1}^{\infty} (1+x^n) = 1 + \sum_{n=1}^{\infty} D(n)x^n$$

This is an example of what's known as a generating function. The function $f(x)$ generates the sequence of numbers $D(n)$, in that they appear as coefficients in its power series expansion. Euler now turned his attention to finding a generating function for the sequence of the number of odd partitions of n , abbreviated by $O(n)$. To construct such a generating function Euler used the same product-expansion principle as before, but this time each of the terms in the product had to contain only odd powers of x , and they needed to be modified to account for possible duplicates. The function $g(x)$ below has the desired property.

$$g(x) = (1+x^1+x^{1+1}+\dots)(1+x^3+x^{3+3}+\dots)(1+x^5+x^{5+5}+\dots)\dots = 1 + \sum_{n=1}^{\infty} O(n)x^n$$

If we assume that $-1 < x < 1$, we may use the geometric series formula to simplify each of the terms in the product and write

$$g(x) = \frac{1}{(1-x)} \cdot \frac{1}{(1-x^3)} \cdot \frac{1}{(1-x^5)} \dots = \prod_{n=1}^{\infty} \frac{1}{(1-x^{2n-1})}$$

At this point, Euler had a startling realization: the generating function $g(x)$ is exactly the same function as $f(x)$. A bit of clever algebra verifies this:

$$f(x) = \prod_{n=1}^{\infty} (1+x^n) = \prod_{n=1}^{\infty} \frac{(1+x^n)(1-x^n)}{(1-x^n)} = \frac{\prod_{n=1}^{\infty} (1-x^{2n})}{\prod_{n=1}^{\infty} (1-x^{2n-1})(1-x^{2n})} = \frac{1}{\prod_{n=1}^{\infty} (1-x^{2n-1})} = g(x)$$

Now since f and g are the same function, they will have the same power series expansions, and thus all of the coefficients in those expansions must agree. But this means that $D(n)=O(n)$ for every integer n , and so the number of distinct partitions is exactly equal to the number of odd partitions for every positive integer! QED.

Euler's proof is remarkable for several reasons. Beyond it being unusually clever and creative, it introduced a whole new way of attacking combinatorial problems: through the use of functions. The infinite sums resulting from expanding f and g have encoded in them the numbers of distinct or odd partitions of all of the positive integers. By studying the single function $f(x)$, we can study the number of distinct partitions for all of the integers at once, and if we happen to prove something about $f(x)$, we may be able to translate that result into a statement about distinct partitions— again, for all of the integers at once. Infinitely many problems may be handled in one stroke. Furthermore, that handling is indirect: notice that nowhere in Euler's proof was it necessary to devise some method of computing the distinct or the odd partitions, or some scheme matching up,

for each integer n , its distinct partitions with its odd ones to show that they are equinumerous. There was no need to find formulas for $D(n)$ and $O(n)$ and then compare them. All that we needed to do was make the observation that the infinite product functions f and g have the coefficients we want if we expand them, and then work with the functions to show they are the same.

This translation of combinatorial problems into analytic problems through the use of generating functions turns out to be extremely useful. Generating functions can be used to find explicit formulas for certain types numbers defined combinatorically or recursively (such as the Catalan numbers or the Fibonacci numbers, for example). They can also be used to give us approximate formulas (in cases where finding explicit formulas is too difficult), recursive relations, or other types of equivalences between the numbers. Partitions is a case in point: to this day nobody knows of an explicit formula for the number of partitions, distinct or not, of a given integer n , but generating functions have been used to find both approximations and further surprising connections between numbers of different types of partitions. Here are a couple examples:

Glaiser's Theorem: For any positive integer n and any integer $d \geq 2$, the number of partitions of n into parts not divisible by d is equal to the number of partitions of n in which no part is repeated d or more times.

Notice that this theorem generalizes Euler's theorem, which is the special case of $d=2$. The proof uses variations of the generating functions Euler had used, but its essence is the same. See [4] for a this and for other similar theorems.

Hardy-Ramanujan Approximation: Let $P(n)$ denote the total number of partitions of a positive integer n . Then

$$P(n) \approx \frac{1}{4\sqrt{3n}} e^{\pi\sqrt{\frac{2n}{3}}}$$

This approximation formula was first obtained in 1918 by G. H. Hardy in collaboration with the brilliant Indian mathematician Srinivasa Ramanujan. Its proof is very difficult, requiring the application of advanced techniques from complex analysis to the generating function for the sequence $P(n)$. The approximation formula is *asymptotic*, meaning the relative error between $P(n)$ and the function on the right approaches zero as n tends to infinity. There is a similar asymptotic formula available for the number of distinct partitions of n :

$$D(n) \approx \frac{1}{4\sqrt[4]{3n^3}} e^{\pi\sqrt{\frac{n}{3}}}$$

We may put the formula to the test using WolframAlpha or another online calculator. If $n=1000$ for example, we find that the function on the right outputs 8.682×10^{21} , while the true number of distinct partitions of 1000 is 8,635,565,795,744,155,161,506, or roughly . The relative error is about 0.5%, so the approximation is already quite good with this moderately-sized value of . Even more accurate approximations are available, both for $P(n)$ and $D(n)$ as well as for other types of partitions too. See [3, pp. 540-547] for the full story.

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String Figures and Thomas F. Storer's Calculus

By: Hillarey Tsui



Introduction

The purpose of this article is to explore the formal mathematical structure of string games.

Did you ever play Cat's Cradle as a child? Many people remember playing games with a loop of string when they were children. Anthropologists call these sorts of games string figures. All over the world people play with loops of string and often create intricate geometric patterns. Making string figures is not just for children, though; there is a thrill and a sense of creativity to it that people of all ages can enjoy. We encourage you to get a long piece of string, tie the ends together, and follow along as we explore the world of string figures.

String figures are typically found in cultures with oral storytelling traditions, and are widespread amongst indigenous peoples around the world¹. Anthropologists have recorded string figures in the Americas, East Africa, Oceania, and Australia. Each string figure is created with distinct features unique to the region where it is made. These features often symbolize an aspect of their community. String figures are used to represent historical events, commonly used tools, and creatures such as butterflies, rabbits, sea gulls, and ducks (Haddon, 1930). Given that the creation of string figures depends on local customs, anthropologists have used string figures to learn about human culture and identify commonalities between cultures.

Mathematicians have long been fascinated by the study of string figures, because of their connections to topology and knot theory. Walter William Rouse Ball (1850-1925) was the first mathematician to take a closer look at the potential connections between string figures and mathematics. In *Mathematical Recreations And Essays* (Nall, 1914) he dedicated a section to the study of string figures and pointed out their connection to knot theory. Ian Stewart, a prolific popularizer of recreational mathematics, also saw a correlation between knot theory, topology, and cat's cradle. In (Stewart, 1997) he writes about "the possibility to create an algebraic expression to get from an initial closed loop of string into interesting shapes using standard moves". Much to Stewart's surprise, Thomas F. Storer (1938-2006), a mathematician at the University of Michigan in Ann Arbor, had already created such a mathematical calculus in the early 1960s. In this article, we'll outline give you a brief introduction to Storer's string figure calculus (Storer, 1988).

What is a String Figure?

A string figure is a pattern made from an unknotted loop of string.

To understand the construction of string figures, we will learn how to represent a figure as a curve in the plane. The outline of a string figure is called its schema. In the language of knot theory, a schema is the planar projection of the string (Adams, 2004). The schema of a string figure also shows the placement of the fingers and positions in three-dimensional space. All schemas consist of arcs of string which meet at simple crossings.

To clarify the exact structure of string positions and crossings, Storer created a local linear description associated with a schema, which he called the "linear sequence" of a figure. The linear sequence is the local description of a string positions and describes the string figure from the perspective of "an ant walking along the string". A linear sequence contains information about the order and type of simple crossings in a schema. See Figure 1 for a simple example of a schema and its linear sequence.

At this point, we encourage the reader to get a loop of string and follow along! Place the

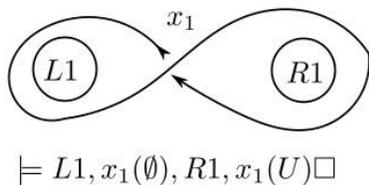


Figure 1: The Infinity Figure

loop around both thumbs, lift the right thumb loop off, and give it a half a twist away from you before placing it back on the right thumb. With any luck you will obtain the figure above.

In this diagram, an over-crossing is denoted by $x_n(\emptyset)$ and an undercrossing by $x_n(u)$. R1 represents the right thumb and L1 represents the left thumb. The linear sequence for this position works like this: $x_1(\emptyset)$ represents an over-crossing as the string passes over itself and $x_1(u)$ represents an under-crossing as the string passes under itself. Because the string forms a closed loop, a linear sequence could in principle anywhere along the string. The fork symbol \boxtimes marks where we choose to begin the linear sequence. Finally, the box denotes the end of the linear sequence.

What is String Figure Calculus?

Storer's functional notation represents the elementary moves performed by the hands on the string as they weave the string into a string figure. In Storer's notation, \square is the left hand and R is the right. All fingers are labelled consecutively from 1 to 5. For example, R2 is the right index finger. Storer's calculus describes the string manipulations as "functors". The fingers which performs a movement is referred to as the "actor". The string that is picked up is the "object" or "argument" of the functor.

In the construction of a string figure the palms of the hands will face one another with the fingers pointing upwards and slightly bent. This is referred to as a normal position and is denoted by (#). When the string is first put on the hands, this is called an opening ,abbreviated $\underline{O}.1$, which is illustrated in Figure 2. To form $\underline{O}.1$: Insert thumbs and little fingers of both hands into a closed loop of string and extend the fingers.

The String Figure Calculus for Opening A

We are now ready to introduce some of the actual moves used in the string figure calculus.

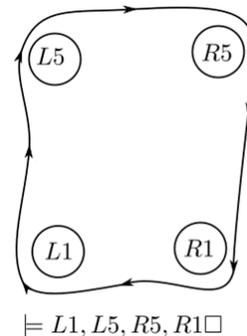


Figure 2: Opening 1

The notation $\overleftarrow{R2}(Lp) \#$ is used to describe the following movement: "The right index finger (R2) reaches to the left, picks up the left palm string (Lp) on its back, and returns to normal position (#)." Notice how the arrow above R2 indicates the direction of travel: the arrow points towards the left and thus R2 moves towards the left.

A loop of string passing around a finger F is denoted by $F\infty$. An arrow that points downwards into a loop means that a finger will go into that loop downwards, as it travels to pick up a string. By combining simple moves to the left and right, with moves through passing loops, one can describe many common manipulations used in making string figures.

The string position below is called Opening A, abbreviated $\underline{O}.A$. It is the opening for the majority of string figures all around the world. Using string figure calculus, it is written:

$$\underline{O}.A \equiv \underline{O}.1 : \overleftarrow{R2}(Lp) \# : \overrightarrow{L2} \downarrow (R2\infty) : \overrightarrow{L2}(Rp) \#$$

This equation asserts that $\underline{O}.A$ is equivalent to $\underline{O}.1$ followed by a sequence of string manipulations. It is a mathematically precise definition of Opening A. . Try it out and see if you can form the figure as shown in Figure 3. See Table 1 for complete step-by-step instructions.

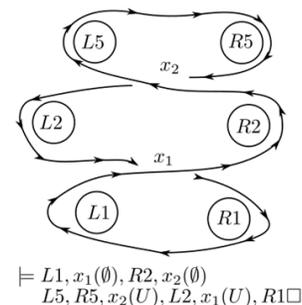
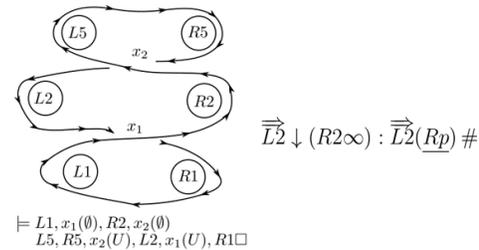
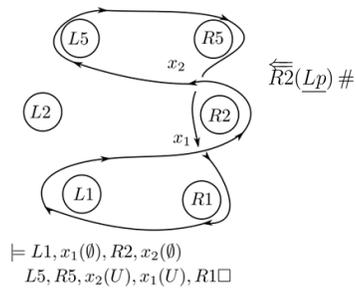
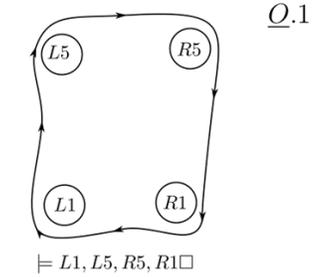


Figure 3: Opening A

The Complete Crossing Analysis for Opening A:



Place the string on the hands in the Opening 1.

Conclusion

In this article, we presented a short introduction to the mathematical structure of string games. If you were able to form Opening A, then have made much progress on learning string figures.

The right index finger to the left, and picks up the left palmar string on its back. Return to normal position.

There are thousands of the string figures in the literature, and we encourage you to investigate further. Camilla Gryski's book *Cat's Cradle, Owl's Eyes*, is a great introduction.

Author Information

This article was prepared by Hillarey Tsui with the help of Parker Glynn-Adey. Parker and Hillarey started working together through the University of Toronto Math Mentorship program, which connects talented highschool students with faculty and graduate students. Hillarey will start her undergraduate studies at the University of Toronto next year. Parker teaches at the University of Toronto Mississauga, where he can often be found playing with string. Feel free to contact him at parker.glynn.adey@utoronto.ca with any questions or comments related to this article.

The left index reaches to the right, descends into the loop surrounding the right index, and picks up the right palm string on its back. Return to normal position.

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MSLC Student Interviews

By: Trevor Cameron

For the Creativity issue of MSLC Magazine, we interviewed students seeking help at the centre and asked them about math, stats, and the way of thinking they promote. We started by asking them if what they have learned in math or stats courses has been useful outside of the classroom:

“Not really. Maybe math, but not so much stats. You can calculate the change you get if it’s not right when you buy something [laughs]. Well, I’ve also read some balance sheets, so those are kind of important.”

We received more than one response like this:

“Yeah, sometimes when I’m working at the cash register and I’m trying to get something done, like calculating customer bills--especially when customers split their bills--it’s good to have those skills. Also, sometimes if you want to average out your total grades it’s good to have some stats in mind [laughs]. Sometimes thinking more logically and quantitatively helps me. It’s good to think less on the emotional side. It’s good to be able to view things in a different way.”

“Math, for sure. I work at Starbucks as a cashier, so all the time [laughs]. Stats, more the conceptual parts of it. Not the calculations, but the way of thinking. In Psych we do a lot of experiments about how different medications can affect you, and in stats we learn about how there are so many factors that can affect things, like sources of error. That way of thinking kind of helps.”

The students seemed to agree that it wasn’t so much the specific mathematical concepts that were useful as the way of thinking they promote. We also asked them if they thought that these skills and this way of thinking might help them in the future:

“When I’m older and I decide to buy a car or a house or make investments, I’m pretty sure knowledge of interest and things like that will be useful.”

“Doing taxes. If I go into graduate school, obviously it’s going to be useful, or if I do a research job, or I guess selling pizzas, you’ve got to know how to calculate the change [laughs].”

Next we asked them if what they’ve learned in math and stats courses has helped them in their other courses:

“Yeah. I was also taking a course in Research Methods in Psychology, and one of the concepts is statistical validity. When you consume a journal article, you’ve also got to be critical, and one of the aspects is to read [and understand] their stats. Even if you don’t do research or go to graduate school, you may need to read a journal article and evaluate the validity of it. It may be useful if I need to read some journal articles.”

“Definitely the problem solving aspect of math. I feel like math and stats make you think in a different way. A lot of math is very systematic.”

“Sometimes, yeah. Some of my other courses are psych-related as well, so this basic knowledge of statistics definitely helps.”

Students also seemed to agree that excelling in math is hard work. We asked them what the most challenging aspect of doing well in math is:

“I guess you just need to do practice questions. It’s not like you just read your textbook and you’re going to learn it all. It really takes practice, and sometimes you just find it very hard to be disciplined and do enough practice questions. It would be nice if you could have a study group or something like that. It’s really not that fun to do stats practice questions instead of watching a Netflix movie, right?”

“You have to practice a lot. I feel that sometimes when you’re not doing as well, you just don’t want to

to practice, because you keep getting it wrong, which makes you not practice, which makes it even worse, so it's kind of like a spiral [laughs]. That's probably the most difficult part--making sure that you reach out for help early on."

Next we asked them if they thought creativity is important in math and stats:

"I do. Math can be very systematic, but sometimes when you're stuck on a problem, and you just think that this is the way and these are the steps, you're just in a little box, but it doesn't bring you to the solution. Being able to think outside of that box is important--is there another way I can conceptualize this problem?"

"Yes, for sure. If there weren't any mathematicians or statisticians, we wouldn't be where we are today. It's definitely good to have an innovative mindset. You never know what might come up. You could come up with a simpler way to approach a math or stats problem."

However, not every student agreed that creativity is important in math:

"Not really [laughs]. I think the impact is more important."

We proceeded to ask this student if it might be different at the graduate level, when you begin to develop your own mathematical theories:

"Maybe when I go to graduate school I'll figure it out then [laughs]."

Well, you can't win 'em all. We were also glad to discover that the students were pleased with the MSLC's services:

"It's fantastic. It's one of the best centres I've ever been to. It's very surprising to me actually, because from my experience, math and stats profs are usually very technical. They're very good at their math stuff, but they're not so good at communication, but the TA's here are really friendly and good at communicating. I guess that's probably because they haven't been ruined by the math [laughs]."

"I think it's great, because stats tutoring sessions are really expensive [laughs]. Having free services available to students who can't even pay tuition, it's great."

We asked them if there were any services not offered at the MSLC that they wished were:

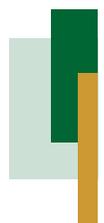
"I guess maybe more practice questions. They could either create more practice questions or lead us through some. At the end of the day, if you don't do practice questions and you're just talking, on the exam, you're by yourself. They can't really help you do the exam, you have to do it yourself, so the practice really makes the difference."

"It might be helpful to do some online services."

Lastly, we asked students if they had any personal stories to share related to their struggles learning math or stats:

"For one assignment in B07 Data Analysis and Statistics in Psychology, I came to the MSLC every day for a whole week, and the MSLC has been really helpful."

"I actually have a very interesting story from my Stats B22 class [Statistics I]. I did really well in my high school stats course. I was attending the first few courses of Statistics I and I thought it was so easy, so I stopped going or doing practice questions, and when I got to the midterm I did really bad. So I learned that just because you did well in your high school stats courses doesn't mean you're going to do well here. Maybe for the first few weeks, which is basically review, but after that, you really need to study hard so you can do well on the midterm. I studied hard after that and I got a good result on the final. So basically, doing well in high school doesn't mean you'll do well in university."





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